ON THE TRANSITIVITY OF PERSPECTIVITY IN CONTINUOUS GEOMETRIES*

BY

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Introduction. The class of finite dimensional projective geometries has been extended to include non-finite dimensional ones by J. von Neumann's remarkable discovery of continuous geometries.‡ In an axiomatic formulation of the geometry as an irreducible complemented modular lattice§ the finiteness of the dimensionality is guaranteed by a chain condition. Von Neumann drops this chain condition and, retaining explicitly only two of its weak consequences, namely, completeness of the geometry and a certain continuity of the lattice operations, succeeds in establishing the existence of an essentially unique real-valued dimension function which may have either a discrete bounded range (the classical finite dimensional projective geometries) or a continuous bounded range (the new continuous geometries). In every case it is understood that the dimension function $D(a)$ is to satisfy

$$D(a+b)+D(ab)=D(a)+D(b) \tag{1}$$

for all a, b.

It is clear that such a dimension function will be closely connected with perspectivities. For a, b are said to be perspective if there exists a c such that

$$a+c=b+c, \qquad ac=bc;$$

and for such a, b (1) implies

$$\begin{aligned} D(a)+D(c) &= D(a+c)+D(ac) \\ &= D(b+c)+D(bc)=D(b)+D(c) \end{aligned}$$

and hence, if $D(c)$ is finite, $D(a)=D(b)$. This motivates a definition of equidimensionality, namely, a and b are called equidimensional if and only if they are perspective. That this definition will lead to the desired dimension function (in an irreducible system) depends in an essential way on the funda-

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‡ See J. von Neumann: (1) Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 92–100, 101–108; (2) *Lectures on Continuous Geometry*, planographed, Institute for Advanced Study, Princeton, N. J., 1935–1937; (3) *Continuous Geometry*, American Mathematical Society Colloquium Lectures, to appear in book form. (2) will be referred to as C.G. The writer wishes to express his thanks to Professor von Neumann for many discussions of his new geometries.

§ See G. Birkhoff, Annals of Mathematics, vol. 36 (1935), pp. 743–748.

mental theorem that a, b equidimensional and b, c equidimensional together imply a, c equidimensional; in other words, that the relation of perspectivity is transitive.

The transitivity of perspectivity has been established by von Neumann for reducible as well as irreducible systems* but partly by indirect methods which require the full force of the completeness and continuity axioms. Now while these axioms are indeed necessary for the existence of the dimension function (in irreducible systems), weaker ones will secure the transitivity of perspectivity (in reducible as well as irreducible systems), in fact, just those parts of von Neumann's axioms which involve at most countable sets of elements.†

The present paper is devoted chiefly to a proof of the transitivity of perspectivity which uses direct methods throughout and holds for all systems satisfying these weaker axioms. The paper is divided into six sections. The weakened set of axioms to be used is formulated in §1. We require parts of C.G., part I, usually in very specialized form, and for convenience these are collected (briefly) in §§2, 3, 4. The new material in the proof of the transitivity of perspectivity is contained in §5. The additivity and continuity properties of perspectivity are established in §6. The Lemma 5.1 in §5 may perhaps be not without some interest of its own.

1. **The partially ordered system.** We shall consider a system L of elements $a, b, c, \cdots, x, y, u, v, \cdots, A, B, \cdots$ which is partially ordered, that is, we shall assume that a relation $a \leqq b$ (written equivalently $b \geqq a$) holds for certain pairs of elements of L in such a way that

(i) $a \leqq b, b \leqq c$ together imply $a \leqq c$, and

(ii) $a \leqq b$, $b \leqq a$ are together equivalent to $a = b$.

The following axioms are postulated:

Axiom I. Countable completeness. *For every finite or countably infinite set‡ of elements $a_1, a_2, \cdots$ there exist the following elements:*

I_1. *a sum element a (written $\sum_n a_n$ or equivalently $a_1 + a_2 + \cdots$) such that for any x of L, $x \geqq a$ if and only if $x \geqq a_n$ for every n,*

I_2. *an intersection element a (written $\prod_n a_n$ or equivalently $a_1 a_2 \cdots$) such that for any x of L, $x \leqq a$ if and only if $x \leqq a_n$ for every n.*

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* For the general case see C.G., part III, p. 22, Theorem 2.3; the special (irreducible) case is also a consequence of the theorems of C.G., part I (see C.G., part I, p. 49, corollary to Theorem 5.16).

† That the "countable" axioms are really weaker than the original axioms of von Neumann can be shown by a simple example which satisfies the "countable" axioms but which has no zero, and hence is not complete.

‡ All sets considered in this paper will be non-void. Thus Axiom I does not imply the existence of a zero or of a unit element.

AXIOM II. COUNTABLE CONTINUITY. *Let $a_1, a_2, \cdots$ be any countably infinite sequence, and let c be an arbitrary element of L. Then*

II$_1$. $(\sum_n a_n)c = \sum_p \{(\sum_{n=1}^p a_n)c\}$;

II$_2$. $(\prod_n a_n) + c = \prod_p \{(\prod_{n=1}^p a_n) + c\}$.

AXIOM III. MODULARITY. *For all a, b, c,*

$$(a + b)c = \{a + (a + c)b\}c,$$

or what is equivalent, $a \leqq c$ implies $(a+b)c = a + bc$.

AXIOM IV. COMPLEMENTATION. *For any three a, b, c such that $a \leqq b \leqq c$ there exists an element d such that $b+d=c$, $bd=a$.*

2. **Independent sets of elements.** We make the following definition:

DEFINITION 2.1. *A finite ($\geqq 2$) or countably infinite set of elements $a_1, a_2, \cdots$ is independent (written $(a_n, n=1, 2, \cdots) \perp$) if for every two mutually exclusive subsets $a_{i_1}, a_{i_2}, \cdots$ and $a_{j_1}, a_{j_2}, \cdots$*

$$\left(\sum_n a_{i_n}\right)\left(\sum_n a_{j_n}\right) = \prod_n a_n.$$

*The a_n are said to be independent over θ if all such $(\sum_n a_{i_n})(\sum_n a_{j_n})$ equal θ.**

LEMMA 2.1. *If the a_n are independent over θ, then $\theta = \prod_n a_n$ and $(a_n, n=1, 2, \cdots) \perp$.*

Proof. Since $\prod_n a_n = a_1(\prod_{n \neq 1} a_n)$, the lemma follows from Definition 2.1.

LEMMA 2.2. *If $a_1, a_2, \cdots$ are independent over θ, then every subset $a_{i_1}, a_{i_2}, \cdots$ is independent over θ.*

Proof. The lemma follows directly from Definition 2.1 and Lemma 2.1.

LEMMA 2.3. *If $a_1, a_2, \cdots$ are independent over θ and if $(a_{i_r}, r=1, \cdots)$ are mutually exclusive subsets for $i=1, 2, \cdots$, then $\sum_r a_{i_r}$, $i=1, 2, \cdots$, are independent over θ.*

Proof. The lemma follows immediately from Definition 2.1.

LEMMA 2.4. *If $\theta, a_1, a_2, \cdots$ are such that for every two finite and mutually exclusive subsets $a_{i_1}, \cdots, a_{i_p}$ and $a_{j_1}, \cdots, a_{j_q}$*

$$\left(\sum_{n=1}^{p} a_{i_n}\right)\left(\sum_{n=1}^{q} a_{j_n}\right) = \theta,$$

then the a_n are independent over θ.

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* If θ is a zero element of L, that is, if $a \geqq \theta$ holds for every a in L, then our independence over θ is precisely the notion of independence as used in C.G., part I, chap. 2.

Proof. Let a_{i_n}, $(n=1, \cdots)$, and a_{j_n}, $(n=1), \cdots$, be any two mutually exclusive subsets of $a_1, a_2, \cdots$. Then

$$\begin{aligned}\Big(\sum_n a_{i_n}\Big)\Big(\sum_n a_{j_n}\Big) &= \Big(\sum_n a_{i_n}\Big)\Big\{\sum_q\Big(\sum_{n=1}^q a_{j_n}\Big)\Big\}\\ &= \sum_q\Big\{\Big(\sum_n a_{i_n}\Big)\Big(\sum_{n=1}^q a_{j_n}\Big)\Big\}\\ &= \cdots = \sum_q\Big\{\sum_p\Big(\sum_{n=1}^p a_{i_n}\Big)\Big(\sum_{n=1}^q a_{j_n}\Big)\Big\}\\ &= \sum_q\Big\{\sum_p(\theta)\Big\} = \theta;\end{aligned}$$

and the lemma follows from Definition 2.1.

COROLLARY. *A countably infinite set of elements is independent over θ if and only if every finite $(\geqq 2)$ subset is independent over θ.*

Proof. The corollary follows immediately from Lemmas 2.2 and 2.4.

LEMMA 2.5. *Let $\theta, a_1, a_2, \cdots$ satisfy $a_n \geqq \theta$ for every n. Let $r_1, r_2, \cdots, r_m$ be distinct integers, and let S be any set of integers not containing r_m. If $a_{r_m}\sum_{n\neq r_m} a_n = \theta$, then*

$$\Big(\sum_{n=1}^m a_{r_n}\Big)\Big(\sum_{n\epsilon S} a_n\Big) = \Big(\sum_{n=1}^{m-1} a_{r_n}\Big)\Big(\sum_{n\epsilon S} a_n\Big).$$

Proof.

$$\begin{aligned}\Big(\sum_{n=1}^m a_{r_n}\Big)\Big(\sum_{n\epsilon S} a_n\Big) &= \Big(\sum_{n=1}^{m-1} a_{r_n} + a_{r_m}\Big)\Big(\sum_{n\neq r_m} a_n\Big)\Big(\sum_{n\epsilon S} a_n\Big)\\ &= \Big\{\Big(\sum_{n=1}^{m-1} a_{r_n}\Big) + a_{r_m}\Big(\sum_{n\neq r_m} a_n\Big)\Big\}\Big(\sum_{n\epsilon S} a_n\Big)\\ &= \Big(\sum_{n=1}^{m-1} a_{r_n} + \theta\Big)\Big(\sum_{n\epsilon S} a_n\Big) = \Big(\sum_{n=1}^{m-1} a_{r_n}\Big)\Big(\sum_{n\epsilon S} a_n\Big),\end{aligned}$$

which proves the lemma.

LEMMA 2.6. *If $\theta, a_1, a_2, \cdots$ are such that $a_{n+1}(a_1 + \cdots + a_n) = \theta$ for every $n=1, 2, \cdots$, then the a_n are independent over θ.*

Proof. By Lemma 2.4 we need only show that

$$\Big(\sum_{n=1}^p a_{i_n}\Big)\Big(\sum_{n=1}^q a_{j_n}\Big) = \theta$$

for all finite p, q and different $i_1, \cdots, i_p; j_1, \cdots, j_q$; and this follows from a finite number of applications of Lemma 2.5.

COROLLARY. *If θ, c, $a_1, a_2, \cdots$ are such that*

$$a_n(a_{n+1} + \cdots + a_{n+p} + c) = \theta$$

for all $n \geqq 1$, $p \geqq 1$, then the c, a_n are independent over θ.

Proof. By Lemma 2.6, $c, a_{n+p}, a_{n+p-1}, \cdots, a_n$ are independent over θ for all $n \geqq 1$, $p \geqq 0$. Therefore, by the corollary to Lemma 2.4, $c, a_1, a_2, \cdots$ are independent over θ.

LEMMA 2.7. *Let $a_1, a_2, \cdots$ be independent over θ. If $S_1, S_2, \cdots$ are arbitrary subsets of the integers $1, 2, \cdots$ and S is the set of the integers common to all S_n, then*

$$\prod_t \Big(\sum_{n \epsilon S_t} a_n \Big) = \sum_{n \epsilon S} a_n.$$

Proof. Let $T = (a_{r_1}, a_{r_2}, \cdots)$ be the set of the integers not in S. Then

$$\begin{aligned} \prod_t \Big(\sum_{n \epsilon S_t} a_n \Big) &= \Big(\sum_{n \epsilon S} a_n + \sum_{n \epsilon T} a_n \Big) \prod_t \Big(\sum_{n \epsilon S_t} a_n \Big) \\ &= \sum_{n \epsilon S} a_n + \Big(\sum_{n \epsilon T} a_n \Big) \prod_t \Big(\sum_{n \epsilon S_t} a_n \Big) \\ &= \sum_{n \epsilon S} a_n + \sum_m \Big\{ \Big(\sum_{t=1}^{m} a_{r_t} \Big) \prod_t \Big(\sum_{n \epsilon S_t} a_n \Big) \Big\} \\ &= \sum_{n \epsilon S} a_n + \sum_m (\theta) = \sum_{n \epsilon S} a_n \end{aligned}$$

(by repeated use of Lemma 2.5) as required.

LEMMA 2.8. *If a_n, $(n = 1, 2, \cdots)$, are independent over θ and $\theta \leqq u_n \leqq a_n$ for $n = 1, \cdots, p$, and if $\theta \leqq v_n \leqq a_n$ for $n = 1, \cdots, q$, then*

$$\Big(\sum_{n=1}^{p} u_n \Big) \Big(\sum_{n=1}^{q} v_n \Big) = \sum_{n=1}^{\min(p,q)} (u_n v_n).$$

Proof.

$$\begin{aligned} (u_1 + u_2)(v_1 + v_2) &= (u_1 + u_2)(u_1 + a_2)(a_1 v_1 + v_2) \\ &= (u_1 + u_2)\{ v_2 + v_1 a_1 (u_1 + a_2) \} \\ &= (u_1 + u_2)\{ v_2 + v_1 (u_1 + a_1 a_2) \} = (u_1 + u_2)(v_2 + v_1 u_1) \\ &= \cdots = (u_1 + u_2)(u_1 v_1 + u_2 v_2) = u_1 v_1 + u_2 v_2. \end{aligned}$$

Thus the lemma holds for $p = q = 2$. But if the lemma holds for all $p = q < m$, then it holds for $p=q=m$ too. For

$$\left(\sum_{n=1}^{m} u_n\right)\left(\sum_{n=1}^{m} v_n\right) = \left(\sum_{n=1}^{m-1} u_n\right)\left(\sum_{n=1}^{m-1} v_n\right) + u_m v_m = \sum_{n=1}^{m} (u_n v_n),$$

since $\theta \leqq \sum_{n=1}^{m-1} u_n$, $\sum_{n=1}^{m-1} v_n \leqq \sum_{n=1}^{m-1} a_n$; $\theta \leqq u_m$, $v_m \leqq a_m$; and $\sum_{n=1}^{m-1} a_n$, a_m are independent over θ. Thus the lemma holds for all $p=q$. If $p \neq q$, say $p<q$, we can set $u_n=\theta$ for $p<n\leqq q$ and apply the result just proved for $p=q$.

LEMMA 2.9. *If $a_1, a_2, \cdots$ are independent over θ and $\theta \leqq u_n$, $v_n \leqq a_n$; for $n=1, 2, \cdots$, then*

$$\left(\sum_{n} u_n\right)\left(\sum_{n} v_n\right) = \sum_{n} (u_n v_n).$$

Proof.

$$\begin{aligned}\left(\sum_{n} u_n\right)\left(\sum_{n} v_n\right) &= \left\{\sum_{p}\left(\sum_{n=1}^{p} u_n\right)\right\}\left\{\sum_{q}\left(\sum_{n=1}^{q} v_n\right)\right\} \\ &= \sum_{q}\left\{\sum_{p}\left(\sum_{n=1}^{p} u_n\right)\left(\sum_{n=1}^{q} v_n\right)\right\} \\ &= \sum_{q}\left\{\sum_{p}\left(\sum_{n=1}^{\min(p,q)} (u_n v_n)\right)\right\} \\ &= \sum_{n} (u_n v_n)\end{aligned}$$

(by Lemma 2.8) as required.

LEMMA 2.10. *Let $a_1, a_2, \cdots$ be independent over θ. If $a_i \geqq a_{ij} \geqq \theta$ for all i, j, and if the elements a_{ij}, $j=1, \cdots$, are independent over θ (whenever there are at least two elements in the set) for every $i=1, 2, \cdots$, then the set of all a_{ij}, $(i, j=1, \cdots)$, is independent over θ.*

Proof. If $a_{i_r j_r}$, $(r=1, \cdots)$, and $a_{k_s l_s}$, $(s=1, \cdots)$, are mutually exclusive subsets of the a_{ij}, then

$$\begin{aligned}\left(\sum_{r} a_{i_r j_r}\right)\left(\sum_{s} a_{k_s l_s}\right) &= \left\{\sum_{i_r}\left(\sum_{j_r} a_{i_r j_r}\right)\right\}\left\{\sum_{k_s}\left(\sum_{l_s} a_{k_s l_s}\right)\right\} \\ &= \sum_{n}\left\{\left(\sum_{j_r} a_{n j_r}\right)\left(\sum_{l_s} a_{n l_s}\right)\right\} \\ &= \sum_{n} \{\theta\} = \theta\end{aligned}$$

(by Lemma 2.9), which proves the lemma.

LEMMA 2.11. *If $\theta \leqq a_0$ and if a_n, a_n' are defined for $n=1, 2, \cdots$ in such a way that*

$$a_{n-1} = a_n + a_n', \qquad a_n a_n' = \theta,$$

for $n=1, 2, \cdots$, then $(\prod_r a_r)$, a_n', $(n=1, 2, \cdots)$, are independent over θ and $a_0 = \sum_n a_n' + \prod_r a_r$.

Proof.

$$a_n'\left(a_{n+1}' + a_{n+2}' + \cdots + a_{n+p}' + \prod_r a_r\right)$$
$$= a_n' a_n\left(a_{n+1}' + a_{n+2}' + \cdots + a_{n+p}' + \prod_r a_r\right) = \theta$$

for all n, p. Hence $(\prod_r a_r)$, a_1', a_2', $\cdots$ are independent over θ by the corollary to Lemma 2.6. Furthermore

$$a_0 = a_1' + a_1 = a_1' + a_2' + a_2 = \cdots$$
$$= \sum_{n=1}^{r} a_n' + a_r = \sum_{n=1}^{\infty} a_n' + a_r,$$

for $r=1, 2, \cdots$. Hence

$$a_0 = \prod_r\left(\sum_{n=1}^{\infty} a_n' + a_r\right) = \sum_{n=1}^{\infty} a_n' + \prod_{r=1}^{\infty} a_r;$$

and the lemma is proved.

3. **Perspectivities and perspective mappings.** We make the following definition:

DEFINITION 3.1. *a, b are perspective (written $a \sim b$) if there exists an element c such that*

(i) $a+c=b+c$,

(ii) $ac=bc$.

Then c is called the axis of the perspectivity.

LEMMA 3.1. *If a, b are perspective and $\theta \leqq ab$, then there exists an element d such that*

(i) $a+d=b+d=a+b$, *and*

(ii) $ad=bd=\theta$.

Proof. Let c be an axis of perspectivity for a and b. Since $\theta \leqq ab \leqq \{c(a+b)+ab\}$, Axiom IV secures the existence of an element d such that

$$d + ab = c(a + b) + ab, \qquad dab = \theta.$$

For this d we have

$$a + d = a + c(a + b) + ab = (a + c)(a + b) = a + b.$$

Similarly $b+d=a+b$, and (i) holds.

$$\begin{aligned} ad &= ad\{c(a + b) + ab\} = d\{ab + ac(a + b)\} \\ &= dab \\ &= \theta, \end{aligned}$$

since $ac(a+b)=ac\leqq ab$. Similarly $bd=\theta$, and (ii) holds. Thus d satisfies the requirements of the lemma.

Definition 3.2. *The sublattice of the x satisfying $x\leqq a$ is denoted by $L(a)$. If $a\leqq b$, the sublattice of the x satisfying $a\leqq x\leqq b$ is denoted by $L(a, b)$.**

Lemma 3.2. *If a, b are perspective with axis c and $\theta=ac=bc$, then a $(1, 1)$ correspondence between the elements of $L(\theta, a)$ and those of $L(\theta, b)$ which preserves the relation $\leqq$ is defined by the inverse mappings*

$$(P) \quad a_1 \rightarrow b_1 = (a_1 + c)b$$

$$(Q) \quad b_1 \rightarrow a_1 = (b_1 + c)a.$$

Proof. If $\theta\leqq x\leqq a$ then under (P) $x\rightarrow(x+c)b$, and under (Q)

$$\begin{aligned} (x + c)b \rightarrow \{(x + c)b + c\}a &= (x + c)(b + c)a \\ &= \{c + x(b + c)\}a = ca + x(b + c) \\ &= x(a + c) = x. \end{aligned}$$

Hence (Q) is inverse to (P). Similarly (P) is inverse to (Q). It follows that the correspondence is $(1, 1)$. The invariance of the relation $\leqq$ is clear from the definition of (P) and (Q).

Definition 3.3. *The mappings of Lemma 3.2 are called perspective mappings.*

Lemma 3.3. *If a_1 corresponds to b_1 under a perspective mapping, then $a_1\sim b_1$.*

Proof. Suppose a_1 corresponds to b_1 under a perspective mapping of $L(\theta, a)$ on $L(\theta, b)$ with axis c. Then a_1 is perspective to b_1 with axis c, for

$$\begin{aligned} a_1 + c &= (b_1 + c)a + c = (b_1 + c)(a + c) \\ &= (b_1 + c)(b + c) = b_1 + c, \end{aligned}$$

$$a_1c = a(b_1 + c)c = ac = bc = b(a_1 + c)c = b_1c,$$

and conditions (i) and (ii) therefore hold.

* The Axioms I, II, III, IV hold in $L(a)$ and in $L(a, b)$. $L(a)$ has a unit (greatest) element, namely a, and $L(a, b)$ has a unit element b and a zero (smallest) element a.

LEMMA 3.4. *If P_i is a perspective mapping of $L(\theta, a_i)$ on $L(\theta, b_i)$ for $i=1, \cdots, p$, where $a_{i+1}=b_i$ for $i=1, \cdots, p-1$, then the product mapping of the P_i is a* (1, 1) *mapping of $L(\theta, a_1)$ on $L(\theta, b_p)$ which preserves the relation $\leqq$.*

Proof. The lemma follows immediately from Definition 3.3.

DEFINITION 3.4. *The mapping of Lemma* 3.4 *is called a projective mapping.*

4. **Transitivity of perspectivity in special cases.** We prove the following lemma:

LEMMA 4.1. *$a\sim b$, $b\sim c$, $(a, b, c) \perp$ together imply $a\sim c$.*

Proof. By Lemma 3.1 x, y exist such that

$$a + x = b + x = a + b, \qquad b + y = c + y = b + c,$$
$$ax = bx = \theta, \qquad by = cy = \theta,$$

where $\theta = abc$.

Then a is perspective to c with axis $d=(a+c)(x+y)$. For

$$\begin{aligned} a + d &= a + (a + c)(x + y) = (a + c)(a + x + y) \\ &= (a + c)(a + b + y) = (a + c)(a + b + c) = a + c. \end{aligned}$$

Similarly $c+d=a+c$. Thus $a+d=c+d$, and (i) holds. Also

$$\begin{aligned} ad &= a(a + c)(x + y) = a(x + y) = a(a + b)(x + y) \\ &= a\{x + (a + b)y(b + c)\} = a\{x + \theta\} = \theta. \end{aligned}$$

Similarly $cd=\theta$. Thus $ad=cd$, and (ii) holds.

LEMMA 4.2. *$a_n\sim b_n$, for $n=1, 2, \cdots$, and $(a_n+b_n, n=1, 2, \cdots) \perp$ together imply $\sum_n a_n\sim\sum_n b_n$.*

Proof. By Lemma 3.1 we may assume the existence of elements x_n such that

$$a_n + x_n = b_n + x_n = a_n + b_n,$$
$$a_n x_n = b_n x_n = \theta,$$

where $\theta=\prod_n(a_n b_n)$. Then $\sum_n a_n$, $\sum_n b_n$ are perspective with axis $\sum_n x_n$, for the relations

$$\begin{aligned} \sum_n a_n + \sum_n x_n &= \sum_n (a_n + x_n) \\ &= \sum_n (b_n + x_n) = \sum_n b_n + \sum_n x_n, \end{aligned}$$

give property (i); and

$$\Big(\sum_n a_n\Big)\Big(\sum_n x_n\Big) = \sum_n (a_n x_n)$$

$$= \sum_n (b_n x_n) = \Big(\sum_n b_n\Big)\Big(\sum_n x_n\Big)$$

(by Lemma 2.9) gives property (ii).

LEMMA 4.3. *If an infinite independent sequence of elements $a_0, a_1, \cdots$ satisfy $a_n \sim a_{n+1}$, for $n=0, 1, \cdots$, then $a_0 = \prod_n a_n$.*

Proof. From Lemmas 2.2 and 4.1, $a_0 \sim a_n$ for all n. By Lemma 3.1 we may therefore assume the existence of elements x_n, $(n=1, 2, \cdots)$, such that

$$a_0 + x_n = a_n + x_n = a_0 + a_n,$$

$$a_0 x_n = a_n x_n = \theta,$$

where $\theta = \prod_n a_n$. We deduce successively

$$a_0 \leqq a_n + x_n, \qquad n = 1, 2, \cdots,$$

$$a_0 \leqq \Big(\sum_{n=p}^{\infty} a_n\Big) + \Big(\sum_{n=1}^{\infty} x_n\Big), \qquad p = 1, 2, \cdots,$$

$$a_0 \leqq \prod_p \Big\{\Big(\sum_{n=p}^{\infty} a_n\Big) + \Big(\sum_{n=1}^{\infty} x_n\Big)\Big\} = \Big\{\prod_p \Big(\sum_{n=p}^{\infty} a_n\Big)\Big\} + \Big(\sum_{n=1}^{\infty} x_n\Big)$$

$$= \sum_{n=1}^{\infty} x_n$$

by Lemma 2.7. Hence

$$a_0 = a_0 \sum_{n=1}^{\infty} x_n = a_0\Big(\sum_p \sum_{n=1}^{p} x_n\Big) = \sum_p \Big(a_0 \sum_{n=1}^{p} x_n\Big) = \theta,$$

if only $a_0 \sum_{n=1}^{p} x_n = \theta$ for $p=1, 2, \cdots$. Now for any fixed p,

$$a_0 \sum_{n=1}^{p} x_n = a_0 \Big\{\sum_{n=1}^{p-1} x_n + \Big(a_0 + \sum_{n=1}^{p-1} x_n\Big) x_p\Big\},$$

and, since

$$x_p\Big(a_0 + \sum_{n=1}^{p-1} x_n\Big) = x_p(a_0 + a_p)\Big(a_0 + \sum_{n=1}^{p-1} a_n\Big)\Big(a_0 + \sum_{n=1}^{p-1} x_n\Big) = x_p a_0 = \theta,$$

therefore

$$a_0 \sum_{n=1}^{p} x_n = a_0 \sum_{n=1}^{p-1} x_n.$$

A finite number of such reductions gives

$$a_0 \sum_{n=1}^{p} x_n = a_0 x_1 = \theta,$$

as required, and the lemma is proved.

LEMMA 4.4. *$a \sim x$, $x \sim a_1$, $ax \leqq a_1 \leqq a$ together imply $a_1 = a$.*

Proof. Let $ax = a_1 x = \theta$. Since $\theta \leqq a_1 \leqq a$, Axiom IV secures the existence of an element a_1' such that

$$a_1 + a_1' = a, \qquad a_1 a_1' = \theta.$$

By Lemmas 3.1 and 3.2 there exist perspective mappings T_1 of $L(\theta, a)$ on $L(\theta, x)$ and T_2 of $L(\theta, x)$ on $L(\theta, a_1)$. Define by induction on n

$$x_n = T_1(a_n), \quad x_n' = T_1(a_n'), \quad a_{n+1} = T_2(x_n), \quad a_{n+1}' = T_2(x_n').$$

Then Lemma 2.11, the relation $ax = \theta$, and Lemma 2.10, give $(a_n', x_n', n = 1, 2, \cdots) \perp$. Lemma 2.2 then gives $(a_n', x_n', a_{n+1}') \perp$, and Lemmas 3.3 and 4.1 give $a_n' \sim a_{n+1}'$, for $n = 1, 2, \cdots$. Lemma 2.2 shows that $(a_n', n = 1, 2, \cdots) \perp$; hence by Lemma 4.3 $a_1' = \prod_n a_n' = \theta$. Thus $a_1 = a_1 + \theta = a_1 + a_1' = a$; and the lemma is proved.

DEFINITION 4.1. *If θ has been defined, we sometimes write*

$$\sum_n (\oplus\, x_n)$$

(or the equivalent $x_1 \oplus x_2 \oplus \cdots$) in place of $\sum_n x_n$ (or $x_1 + x_2 + \cdots$), provided the x_n are independent over θ. If $\theta \leqq u \leqq v$, then $[v - u]$ will denote an element (fixed) such that $u \oplus [v - u] = v$. (Such an element exists by Axiom IV.)

LEMMA 4.5. *$a \sim x$, $x \sim b$, $ab \leqq x$ together imply $a \sim b$.*

Proof. (α) Consider the special case where $ab = bx = ax$ and $x \leqq a + b$. Let $b_1 = b(a + x)$. Then b_1 is perspective to x with axis a, for

$$b_1 + a = b(a + x) + a = (b + a)(a + x) = x + a,$$

hence relation (i) holds; and

$$b_1 a = b(a + x)a = ba = xa,$$

hence (ii) holds.

Since $b \sim x$ and $b_1 x = b(a + x)x = bx$, $bx \leqq b_1 \leqq b$; and Lemma 4.4 gives $b_1 = b$. Hence $b \leqq a + x$. Similarly $a \leqq b + x$. It follows that a is perspective to b with axis x, for

$$a + x = a + x + b = b + x;$$

hence relation (i) holds, and $ax = bx$ (by the special hypotheses of (α)) hence relation (ii) holds. This proves Lemma 4.5 in the special case (α).

(β) Suppose now only that $ab = bx = ax$ (equal, say θ). Let T_1, T_2 be perspective mappings of $L(\theta, x)$ on $L(\theta, a)$ and on $L(\theta, b)$, respectively. Set $a_0 = a$, $x_0 = x$, $b_0 = b$, and define a_n, a_n', x_n, x_n', b_n, b_n', for $n = 1, 2, \cdots$, by induction on n as follows:

$$\begin{aligned} x_n &= x_{n-1}(a_{n-1} + b_{n-1}), & a_n &= T_1(x_n), & b_n &= T_2(x_n),\\ x_n' &= [x_{n-1} - x_n], & a_n' &= T_1(x_n'), & b_n' &= T_2(x_n'). \end{aligned}$$

If we set $\bar{a} = \prod_n a_n$, $\bar{x} = \prod_n x_n$, $\bar{b} = \prod_n b_n$, then Lemma 2.11, the relation $ab = \theta$, and Lemma 2.3 give $a = \bar{a} + \sum_n a_n'$, $b = \bar{b} + \sum_n b_n'$; and $\bar{a} + \bar{b}$, $a_n' + b_n'$, $(n = 1, 2, \cdots)$, are independent over θ. Lemma 3.3 shows that $a_n' \sim x_n'$ and $x_n' \sim b_n'$. Since $a_n' b_n' = \theta$ and $x_n'(a_n' + b_n') = \theta$, Lemmas 2.6 and 4.1 show that $a_n' \sim b_n'$, for $n = 1, 2, \cdots$. Now Lemma 4.2 shows that $a \sim b$ if only $\bar{a} \sim \bar{b}$.

$\bar{a}$, $\bar{x}$, $\bar{b}$ satisfy the hypotheses of Lemma 4.5 and the special conditions of (α), for $\bar{a} = T_1(\bar{x})$, $\bar{b} = T_2(\bar{x})$ imply $\bar{a} \sim \bar{x}$, $\bar{x} \sim \bar{b}$; $\bar{a}\bar{x} = \bar{a}ax\bar{x} = \theta$, $\bar{b}\bar{x} = \bar{b}bx\bar{x} = \theta$, $\bar{a}\bar{b} = \bar{a}ab\bar{b} = \theta$; and, since $x_n \leqq a_{n-1} + b_{n-1}$ for all n,

$$\begin{aligned} \bar{x} \leqq \prod_n (a_n + b_n) &= \prod_q \prod_p \left(\prod_{n=1}^{p} a_n + \prod_{n=1}^{q} b_n \right) \\ &= \prod_n a_n + \prod_n b_n = \bar{a} + \bar{b} \end{aligned}$$

by two applications of Axiom II_2. Hence $\bar{a} \sim \bar{b}$; and Lemma 4.5 is proved for the special case (β).

(γ) Suppose now only $ab = bx$ (equal, say θ). Let T_1, T_2 be perspective mappings of $L(\theta, a)$ on $L(\theta, x)$ and of $L(\theta, x)$ on $L(\theta, b)$, respectively. Set

$$\begin{aligned} a_1 &= ax, & x_1 &= T_1(a_1) = a_1, & b_1 &= T_2(x_1),\\ a_2 &= [a - ax], & x_2 &= T_1(a_2), & b_2 &= T_2(x_2). \end{aligned}$$

Then $a = a_1 \oplus a_2$, $x = x_1 \oplus x_2$, $b = b_1 \oplus b_2$. By Lemma 3.3, $a_1 = x_1 \sim b_1$. Since the hypotheses of Lemma 4.5 and the special conditions of (β) are clearly satisfied by a_2, x_2, b_2, $a_2 \sim b_2$. Lemma 4.2 now gives $a \sim b$, and Lemma 4.5 is established for the special case (γ).

(δ) Suppose finally only the hypotheses of Lemma 4.5. The method by which (γ) was deduced from (β) can be applied in the same way to deduce (δ) from (γ). Thus Lemma 4.5 is proved.

COROLLARY. *If T_1, T_2 are perspective mappings of $L(\theta, a)$ on $L(\theta, x)$, and of $L(\theta, x)$ on $L(\theta, b)$, respectively, and if T_2T_1 maps a_1 on b_1, then $a_1b_1 = \theta$ implies $a_1 \sim b_1$.*

Proof. Set $x_1=T_1(a_1)$. Lemmas 3.3 and 4.5 applied to a_1, x_1, b_1, give the desired result.

5. **Transitivity of perspectivity.** We prove the following lemma:

LEMMA 5.1. *If $a_1 \geqq a_2 \geqq \cdots$ and c are given, and if $\theta = \prod_n (a_n c)$, then there exist decompositions*

$$a_n = a_n c \oplus a_n' ,$$

for $n=1, 2, \cdots$, such that $a_1' \geqq a_2' \geqq \cdots$.

Proof. Let $I_n = [a_n - (a_n c + a_{n+1})]$, for $n=1, 2, \cdots$, and let $\bar{a} = \prod_n a_n$, $\bar{I} = [\bar{a} - \bar{a}c]$. Then

$$\begin{aligned} a_1 &= I_1 \oplus (a_1 c + a_2), \\ a_2 &= I_2 \oplus (a_2 c + a_3), \\ &\cdots\cdots\cdots, \\ a_n &= I_n \oplus (a_n c + a_{n+1}), \\ &\cdots\cdots\cdots, \\ \bar{a} &= \bar{I} \oplus \bar{a}c; \end{aligned}$$

and clearly $c\bar{I} = c\bar{I}\bar{a}c = \theta$, $cI_r = cI_r(a_r c + a_{r+1}) = \theta$, $I_{r+k} \leqq a_{r+k} \leqq a_r$, for all $r \geqq 1$, $k \geqq 0$.

We can now prove that $a_r = a_r c \oplus \bar{I} \oplus \sum_{n=r}^{\infty} I_n$, for $r=1, 2, \cdots$. In the first place, $a_r c$, $\bar{I}$, I_n, $(n=r, r+1, \cdots)$ are independent over θ by the corollary to Lemma 2.6 since $a_r c\bar{I} = \theta$; and for all $p > n \geqq r$,

$$\begin{aligned} I_n\left(a_r c + \bar{I} + \sum_{m=n+1}^{p} I_m\right) &= I_n a_n (a_r c + a_{n+1})\left(a_r c + \bar{I} + \sum_{m=n+1}^{p} I_m\right) \\ &= I_n (a_n c + a_{n+1})\left(a_r c + \bar{I} + \sum_{m=n+1}^{p} I_m\right) \\ &= \theta. \end{aligned}$$

Secondly,

$$\begin{aligned} a_r c + \bar{I} + \sum_{n=r}^{\infty} I_n &= a_r c + \prod_{m=r}^{\infty} a_m + \sum_{n=r}^{\infty} I_n \\ &= \prod_{m=r}^{\infty}\left(a_r c + a_m + \sum_{n=r}^{\infty} I_n\right) \\ &= \prod_{m=r}^{\infty}\left(a_r c + a_m + \sum_{n=r}^{m-1} I_n\right). \end{aligned}$$

Now if $m \geqq r$,

$$a_r c + a_m + \sum_{n=r}^{m-1} I_n = a_r c + a_{m-1} c + a_m + I_{m-1} + \sum_{n=r}^{m-2} I_n$$
$$= a_r c + a_{m-1} + \sum_{n=r}^{m-2} I_n = \cdots$$
$$= a_r c + a_r = a_r.$$

Thus $a_r c \oplus \bar{I} \oplus \sum_{n=r}^{\infty} I_n = \prod_{m=r}^{\infty}(a_r) = a_r$ as required. It is now clear that if we set

$$a_n' = \bar{I} \oplus \sum_{r=n}^{\infty} I_r,$$

we will obtain the decompositions required by the lemma.

COROLLARY. *In Lemma* 5.1, $\bar{I}$, I_n, $(n=1, 2, \cdots)$, *are independent over* θ.

Proof. This is immediate from the proof of Lemma 5.1.

THEOREM 5.1. TRANSITIVITY OF PERSPECTIVITY. $a\sim x$, $x\sim b$ *together imply* $a\sim b$.

Proof. (I) Let $\theta = abx$, and let T_1, T_2 be perspective mappings of $L(\theta, a)$ on $L(\theta, x)$ and of $L(\theta, x)$ on $L(\theta, b)$, respectively. Let $T = T_2T_1$ be the product mapping of $L(\theta, a)$ on $L(\theta, b)$, and let T^{-1} be the inverse mapping to T. We shall use the notation $a_1 \rightarrow b_1$, or the equivalent $b_1 = T(a_1)$, to denote that b_1 is the map of a_1 under T.

(II) Let $c = ab$, and let $a_0 = [a - c]$, where a_0 is restricted to satisfy a certain condition which will be stated precisely later (see (III) below). Set $b_1 = (T(a_0))c$, $b_1' = [T(a_0) - b_1]$, $a_1 = T^{-1}(b_1)$, $a_1' = T^{-1}(b_1')$. Then

$$a = a_0 \oplus c,$$
$$a_0 = a_1 \oplus a_1', \qquad a_1 \rightarrow b_1 \leqq c, \qquad a_1' \rightarrow b_1', \qquad (b_1' c = \theta).$$

Since $\theta \leqq b_1 \leqq a$, $T(b_1)$ is defined. Let $\{T(b_1)\}c = b_2$, $b_2' = [T(b_1) - b_2]$, $b_{12} = T^{-1}(b_2)$, $a_2 = T^{-1}(b_{12})$, $b_{12}' = T^{-1}(b_2')$, $a_2' = T^{-1}(b_{12}')$. Then

$$a_1 = a_2 \oplus a_2', \qquad a_2 \rightarrow b_{12} \rightarrow b_2 \leqq c, \qquad a_2' \rightarrow b_{12}' \rightarrow b_2', \qquad (b_2' c = \theta).$$

Similarly, obtain the table

$$\begin{array}{llll}
a_0 = a_1 \oplus a_1', & a_1 \rightarrow b_1 \leqq c, & a_1' \rightarrow b_1', & (b_1' c = \theta), \\
a_1 = a_2 \oplus a_2', & a_2 \rightarrow b_{12} \rightarrow b_2 \leqq c, & a_2' \rightarrow b_{12}' \rightarrow b_2', & (b_2' c = \theta), \\
\cdots\cdots & \cdots\cdots & \cdots\cdots, & \\
a_{n-1} = a_n \oplus a_n', & a_n \rightarrow b_{12\cdots n} \rightarrow b_{23\cdots n} \rightarrow \cdots \rightarrow b_n \leqq c, & & \\
 & a_n' \rightarrow b_{12\cdots n}' \rightarrow b_{23\cdots n}' \rightarrow \cdots \rightarrow b_n', & & (b_n' c = \theta), \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & .
\end{array}$$

We can now prove the following statements:

(α) *$\prod_{n=1}^{\infty} a_n=\theta$ and $a_0=\sum_{n=1}^{\infty}(\oplus a_n')$.*

(β) *a_n', $b_{12\cdots n}'$, $b_{23\cdots n}'$, $\cdots$, b_n' are independent over θ, for $n=1, 2, \cdots$.*

(γ) *If we set $d_n=a_n'+b_{12\cdots n}'+b_{23\cdots n}'+\cdots+b_n'$, then $d_1, d_2, \cdots$ are independent over θ.*

(δ) *All the primed elements a_1', a_2', $\cdots$, b_1', b_{12}', $\cdots$ are independent over θ.*

Proof of (α). Let $\prod_n a_n=\bar{a}$. Then $T^n(\bar{a})=TT\cdots T(\bar{a})$ (n factors T) is defined for $n=1, 2, \cdots$; and $T(\bar{a}), T^2(\bar{a}), \cdots$ are independent over θ by the corollary to Lemma 2.6 since

$$\{T^n(\bar{a})\}\{T^{n+1}(\bar{a})+\cdots+T^{n+p}(\bar{a})\}$$

has a T^{-n} map which is $\leqq a_0 b=a_0ab=a_0c=\theta$, for all n, $p\geqq 1$. Since $T^n(\bar{a})\sim T^{n+1}(\bar{a})$ by Lemma 3.3, Lemma 4.3 shows that $\bar{a}=\theta$. The statement (α) now follows from Lemma 2.11.

Proof of (β).

$$b_{r(r+1)\cdots n}'(b_{(r+1)(r+2)\cdots n}'+b_{(r+2)(r+3)\cdots n}'+\cdots+b_n')=\theta$$

since it has a T^{-r} map which is $\leqq a_0b=\theta$. The statement (β) now follows from the corollary to Lemma 2.6.

Proof of (γ). $\theta\leqq d_n(d_{n+1}+d_{n+2}+\cdots+d_{n+p})$

$$\leqq (a_n'+b_{12\cdots n}'+\cdots+b_n')\left\{\begin{array}{l} a_{n+1}'+b_{12\cdots(n+1)}'+\cdots+b_{n+1}' \\ +a_{n+2}'+b_{12\cdots(n+2)}'+\cdots+b_{n+2}' \\ +\cdots\cdots\cdots\cdots\cdots \\ +a_{n+p}'+b_{12\cdots(n+p)}'+\cdots+b_{n+p}' \end{array}\right\}$$

$$= a_n'(a_{n+1}'+a_{n+2}'+\cdots+a_{n+p}')+(b_{12\cdots n}'+\cdots+b_n')\left\{\begin{array}{l} b_{12\cdots(n+1)}'+\cdots+b_{n+1}' \\ +b_{12\cdots(n+2)}'+\cdots+b_{n+2}' \\ +\cdots\cdots\cdots\cdots \\ +b_{12\cdots(n+p)}'+\cdots+b_{n+p}' \end{array}\right\}$$

$$=\theta+T\left[(a_n+b_{12\cdots n}+\cdots+b_{(n-1)n})\left\{\begin{array}{l} a_{n+1}'+b_{12\cdots(n+1)}'+\cdots+b_{n(n+1)}' \\ +a_{n+2}'+b_{12\cdots(n+2)}'+\cdots+b_{(n+1)(n+2)}' \\ +\cdots\cdots\cdots\cdots\cdots\cdots \\ +a_{n+p}'+b_{12\cdots(n+p)}'+\cdots+b_{(n+p-1)(n+p)}' \end{array}\right\}\right]$$

$$\leqq\cdots\leqq T^{n+1}(\theta)=\theta.$$

The statement (γ) now follows from the corollary to Lemma 2.6.

Proof of (δ). The statement (δ) follows from (β) and (γ) by Lemma 2.10.

(III) By the corollary to Lemma 4.5 $a_n'\sim b_{12\cdots n}'$, $b_{12\cdots n}'\sim b_{23\cdots n}'$, $\cdots$, $b_{(n-1)n}\sim b_n'$. Since a_n', $b_{12\cdots n}'$, $\cdots$, b_n' are independent, repeated application of Lemma 4.1 shows that $a_n' \sim b_n'$ for $n=1, 2, \cdots$. Now set

$$A = \sum_{n=1}^{\infty} (\oplus a_n') \oplus \sum_{r \neq s} (\oplus b'_{r(r+1)\cdots s}),$$

$$B = \sum_{n=1}^{\infty} (\oplus b_n') \oplus \sum_{r \neq s} (\oplus b'_{r(r+1)\cdots s}).$$

By Lemma 4.2, $A \sim B$. Furthermore $T(A) = B$.

Now suppose that a_0 was chosen (in (II) above) in such a way that, the b_n' having been defined as above, we should have $(\sum_{n=1}^{\infty} b_n')c = \theta$ (that such an a_0 exists will be shown in (V) below). Setting

$$g = \left[c - \sum_{r \neq s} (\oplus b'_{r(r+1)\cdots s}) \right]$$

we have $Bg = Bgc = g \sum_{r \neq s}(\oplus b'_{r(r+1)\cdots s}) = \theta$, and we can define

$$h = [b - (B \oplus g)].$$

Then we clearly have

$$c = \sum_{r \neq s} (\oplus b'_{r(r+1)\cdots s}) \oplus g,$$

$$a = A \oplus g,$$

$$b = B \oplus g \oplus h.$$

By Lemma 4.2, $a \sim b$ if only $h = \theta$.

(IV) We proceed to show that $h = \theta$. Let $g' = T^{-1}(g + h)$. Since

$$A \oplus g = a = T^{-1}(b) = T^{-1}(B \oplus g \oplus h) = A \oplus g',$$

it follows that there exists a perspective mapping S of $L(\theta, g')$ on $L(\theta, g)$. Now set $h_0 = h$ and define h_n', h_n, for $n = 1, 2, \cdots$, by induction on n as follows:

$$h_n' = T^{-1}(h_{n-1}), \qquad h_n = S(h_n').$$

Then $h_0, h_1, \cdots$ are independent over θ by the corollary to Lemma 2.6 since

$$\begin{aligned} h_n(h_{n+1} + \cdots + h_{n+p}) &= (ST^{-1})^n\{h(h_1 + \cdots + h_p)\} \\ &= (ST^{-1})^n\{hg(h_1 + \cdots + h_p)\} \\ &= (ST^{-1})^n(\theta) = \theta. \end{aligned}$$

Since $h_{n-1} \sim h_n'$, $h_n' \sim h_n$, and $h_{n-1}h_n = \theta \leqq h_n'$, Lemma 4.5 shows that $h_{n-1} \sim h_n$. Now Lemma 4.3 gives $h = h_0 = \theta$ as required.

(V) To complete the proof of Theorem 5.1 we need only show that the $a_0=[a-ab]=[a-c]$ defined in (II) above could be chosen in such a way that $c\sum_{n=1}^{\infty}b_n'=\theta$ will hold in (III) above. We first note that if we set

$$v_1 = T^{-1}(c),\ v_2 = T^{-1}(v_1c),\ \cdots,\ v_{n+1} = T^{-1}(v_nc),\ \cdots,$$

then it is sufficient to choose a_0 so that $a=a_0\oplus c$ and

$$v_nc = (v_n + a_0)c,$$

for $n=1, 2, \cdots$. For if a_0 is so chosen, then

$$\begin{aligned}
\theta &\leqq b_{n+1}'(c + b_1' + b_2' + \cdots + b_n') \\
&= T\{b_{n(n+1)}'(T^{-1}(c) + a_1' + b_{12}' + \cdots + b_{(n-1)n}')\} \\
&\leqq T\{b_{n(n+1)}'c(v_1 + a_0 + b_{12}' + \cdots + b_{(n-1)n}')\} \\
&= T\{b_{n(n+1)}'(c(v_1 + a_0) + b_{12}' + \cdots + b_{(n-1)n}')\} \\
&= T\{b_{n(n+1)}'(v_1c + b_{12}' + \cdots + b_{(n-1)n}')\} \\
&\leqq T^2\{b_{(n-1)n(n+1)}'(v_2 + a_0 + b_{123}' + \cdots + b_{(n-2)(n-1)n}')\} \\
&\leqq T^2\{b_{(n-1)n(n+1)}'(v_2c + b_{123}' + \cdots + b_{(n-2)(n-1)n}')\} \\
&\leqq \cdots \leqq T^n(b_{12\cdots(n+1)}'v_nc) \leqq b_{n+1}'c = \theta;
\end{aligned}$$

hence

$$b_{n+1}'(c + b_1' + b_2' + \cdots + b_n') = \theta,$$

for $n=1, 2, \cdots$. By Lemma 2.6, $c, b_1', b_2', \cdots$ are independent over θ; hence $c\sum_{n=1}^{\infty}b_n'=\theta$ as required.

Thus we have only to construct an a_0 such that

$$a=a_0 \oplus c, \quad v_nc = (v_n + a_0)c,$$

for $n=1, 2, \cdots$. Apply Lemma 5.1 to $v_1\geqq v_2\geqq \cdots$ and c, and obtain $\bar{I}, I_n$, $n=1, 2, \cdots$, as in the proof of Lemma 5.1. Then $v_n=v_nc\oplus\bar{I}\oplus\sum_{m=n}^{\infty}I_m$. Let $u=[a-(v_1+c)]$. Then

$$u\left(\bar{I} + \sum_{m=1}^{\infty} I_m\right) = uv_1\left(\bar{I} + \sum_{m=1}^{\infty} I_m\right) = \theta,$$

and we can set

$$a_0 = \bar{I} \oplus \sum_{m=1}^{\infty} (\oplus I_m) \oplus u.$$

This a_0 satisfies our requirements, for

$$a_0 + c = c + \bar{I} + \sum_{m=1}^{\infty} I_m + u = c + v_1 + u = a,$$

$$a_0c = a_0(v_1 + c)c = \left(I + \sum_{m=1}^{\infty} I_m\right)c$$
$$= \left(I + \sum_{m=1}^{\infty} I_m\right)v_1c = \theta.$$

Hence $a_0 \oplus c = a$. And

$$(v_n + a_0)c = \left(v_nc + I + \sum_{m=1}^{\infty} I_m + u\right)c$$
$$= (v_nc + a_0)c$$
$$= v_nc + a_0c = v_nc + \theta = v_nc.$$

This completes the proof of Theorem 5.1.

6. **Additivity and continuity properties of perspectivity.** We prove the following lemma:

Lemma 6.1. *If θ is defined and if*

$$a = a_1 \oplus a_1' = a_2 \oplus a_2',$$

then $a_1 \sim a_2$ implies $a_1' \sim a_2'$.

Proof. The perspectivity $a_1 \sim a_2$ implies, by Lemma 3.1, the existence of an x for which

$$a_1 + x = a_2 + x = a_1 + a_2,$$
$$a_1x = a_2x = \theta.$$

Let $c = [a - (a_1 + a_2)]$. Then a_1' is perspective to $(x + c)$ with axis a_1, for we have the relation

$$a_1' + a_1 = a = c + (a_1 + a_2) = c + (a_1 + x)$$
$$= (x + c) + a_1,$$

hence relation (i) holds; and

$$a_1' a_1 = \theta = xa_1 = (x + \theta)a_1 = \left\{x + (a_1 + a_2)c\right\}a_1$$
$$= \left\{x + (a_1 + x)c\right\}a_1 = (x + c)a_1,$$

hence relation (ii) holds.

Similarly a_2' is perspective to $(x + c)$. Theorem 5.1 then proves that $a_1' \sim a_2'$ as required.

Lemma 6.2. *If θ is defined, and if*

$$a = a_1 \oplus a_1', \qquad b = b_1 \oplus b_1',$$

then $a \sim b$, $a_1 \sim b_1$ together imply $a_1' \sim b_1'$.

Proof. Let T be a perspective mapping of $L(\theta, a)$ on $L(\theta, b)$, and let $u = T(a_1)$, $v = T(a_1')$; then $a_1 \sim u$, $a_1' \sim v$, and $b = u \oplus v$. Since $a_1 \sim b_1$, Theorem 5.1 gives $u \sim b_1$ and Lemma 6.1 gives $v \sim b_1'$. Since $a_1' \sim v$ and $v \sim b_1'$, Theorem 5.1 gives $a_1' \sim b_1'$.

LEMMA 6.3. *$a_1 \sim b_1$, $a_2 \sim b_2$, $a_1a_2 = b_1b_2$ together imply $(a_1 + a_2) \sim (b_1 + b_2)$.*

Proof. Let $\theta = a_1a_2 = b_1b_2$, $d = a_1 + a_2 + b_1 + b_2$, and define $a' = [d - (a_1 \oplus a_2)]$, $b' = [d - (b_1 \oplus b_2)]$. Then

$$(a' \oplus a_2) \oplus a_1 = d = (b' \oplus b_2) \oplus b_1.$$

By Lemma 6.1, $(a' \oplus a_2) \sim (b' \oplus b_2)$; hence by Lemma 6.2 $a' \sim b'$. Since $a' \oplus (a_1 \oplus a_2) = b' \oplus (b_1 \oplus b_2)$ Lemma 6.1 proves $(a_1 + a_2) \sim (b_1 + b_2)$ as required.

LEMMA 6.4. *If θ is defined, and if $a_1, \cdots, a_p$ and $b_1, \cdots, b_p$ are two sets of elements, each independent over θ, with $a_r \sim b_r$ for $r = 1, \cdots, p$, where $p = 1, 2, \cdots$, then*

$$\left(\sum_{r=1}^{p} (\oplus\, a_r)\right) \sim \left(\sum_{r=1}^{p} (\oplus\, b_r)\right).$$

Proof. Suppose the lemma established for $p = n$ for some fixed $n = 1, 2, \cdots$. Then

$$(a_1 \oplus \cdots \oplus a_n) \sim (b_1 \oplus \cdots \oplus b_n), \qquad a_{n+1} \sim b_{n+1}$$

imply, by Lemma 6.3, $(a_1 + \cdots + a_n + a_{n+1}) \sim (b_1 + \cdots + b_n + b_{n+1})$; and the lemma will hold for $p = n + 1$. Since the lemma is trivially true for $p = 1$, it holds, by induction, for all $p = 1, 2, \cdots$.

We now define a relation $a \propto b$ as follows:

DEFINITION 6.1. *$a \propto b$ if $a \sim b_1$ for some $b_1 \leqq b$.*

LEMMA 6.5. (I) *$a \leqq b$ implies $a \propto b$.*
(II) *$a \leqq b$, $b \propto c$ together imply $a \propto c$.*
(III) *$a \propto b$, $b \propto c$ together imply $a \propto c$.*
(IV) *$a \propto b$, $b \propto a$ together imply $a \sim b$.*

Proof. (I) follows from $a \sim a$.

(II): Let $\theta = ac$, and let T be a perspective mapping of $L(\theta, b)$ on $L(\theta, c)$. Then $T(a)$ is defined, $a \sim T(a)$, and $T(a) \leqq c$. Hence $a \propto c$.

(III): $a \propto b$ means $a \sim b_1$ for some $b_1 \leqq b$. Since $b \propto c$, (II) gives $b_1 \sim c_1$ for some $c_1 \leqq c$. Theorem 5.1 then gives $a \sim c_1$. Hence $a \propto c$ as required.

(IV): $a \propto b$ means $a \sim b_1$ for some $b_1 \leqq b$. If $b \propto a$, then $b \sim a_1$ for some $a_1 \leqq a$, and $b_1 \sim a_2$ for some $a_2 \leqq a_1$, by (II). Then, by Theorem 5.1, $a \sim a_2$; and, since $a_2 \leqq a$, Lemma 4.4 implies $a_2 = a$. Since $a_2 \leqq a_1 \leqq a$, we have $a = a_1 \sim b$; that is $a \sim b$ as required.

LEMMA 6.6. *If $c \propto a$ and $d = ca$, there exists an element a_1 with $d \leqq a_1 \leqq a$ and $c \sim a_1$.*

Proof. $c \propto a$ means $c \sim a_2$ for some $a_2 \leqq a$. Let $\theta = a_2 d$; then $\theta \leqq ca_2$, and there exists, by Lemmas 3.1 and 3.2, a perspective mapping T of $L(\theta, c)$ on $L(\theta, a_2)$. Define $c' = [c - d]$ and $\bar{c}' = T(c')$; then $\bar{c}' \leqq a_2 \leqq a$, $c'd = \theta$, $c' \sim \bar{c}'$, and $\bar{c}'d = \bar{c}'a_2 d = \theta$. By Lemma 6.3,

$$c = (c' \oplus d) \sim (\bar{c}' \oplus d) \leqq a,$$

and $a_1 = \bar{c}' \oplus d$ satisfies all the requirements of the lemma.

LEMMA 6.7. *If θ is defined and if $a \oplus a_1 \geqq a' \oplus a_2$, then $a \sim a'$ implies $a_2 \propto a_1$.*

Proof. Let $v = [(a \oplus a_1) - (a' \oplus a_2)]$. Then $a \oplus a_1 = a' \oplus a_2 \oplus v$, and Lemma 6.1 implies $a_1 \sim (a_2 \oplus v)$; thus $a_2 \propto a_1$ as required.

LEMMA 6.8. *If θ is defined and if $a \oplus a_2 \propto a \oplus a_1$, then $a_2 \propto a_1$.*

Proof. By Lemma 6.6 (with the θ of the present lemma in the place of the d of Lemma 6.6) $(a \oplus a_2) \sim u$, where $\theta \leqq u \leqq (a + a_1)$. Let T be a perspective mapping of $L(\theta, a + a_2)$ on $L(\theta, u)$, and let $\bar{a} = T(a)$ and $\bar{a}_2 = T(a_2)$; then $a \oplus a_1 \geqq \bar{a} \oplus \bar{a}_2$ and $a \sim \bar{a}$. By Lemma 6.7, $a_2 \propto a_1$, and since $\bar{a}_2 \sim a_2$, Lemma 6.5 (III) implies $a_2 \propto a_1$, which proves the lemma.

LEMMA 6.9. *If θ is defined, and if*

$$u \oplus v_1 \oplus \cdots \oplus v_p \propto a,$$

where $u \leqq a$ and $(v_1 \oplus \cdots \oplus v_p)a = \theta$, $(p = 1, 2, \cdots)$, then there exist elements $v_1', \cdots, v_p'$ with $v_r \sim v_r'$, $(r = 1, \cdots, p)$, $u, v_1, \cdots, v_p, v_1', \cdots, v_p'$ independent over θ, and $u \oplus v_1' \oplus \cdots \oplus v_p' \leqq a$.

Proof. Define $a_1 = [a - u]$; then

$$a = u \oplus a_1, u \oplus (v_1 \oplus \cdots \oplus v_p) \propto u \oplus a_1,$$

and Lemma 6.8 implies $(v_1 \oplus \cdots \oplus v_p) \propto a_1$. Thus there exists, by Lemma 6.6, a perspective mapping T of $L(\theta, (v_1 \oplus \cdots \oplus v_p))$ on $L(\theta, a_2)$ for some a_2 satisfying $\theta \leqq a_2 \leqq a_1$. Let $v_r' = T(v_r)$ for $r = 1, \cdots, p$. Then $v_r \sim v_r'$, for $r = 1, \cdots, p$; $\sum_{r=1}^{p}(\oplus v_r') \leqq a_2 \leqq a_1 \leqq a$; and v_r', $(r = 1, \cdots, p)$, are independent over θ. Since

$$\left(\sum_{r=1}^{p}(\oplus v_r')\right)\left(u \oplus \sum_{r=1}^{p}(\oplus v_r)\right) = \left(\sum_{r=1}^{p}(\oplus v_r')\right)a_1 a\left(u \oplus \sum_{r=1}^{p}(\oplus v_r)\right)$$
$$= \left(\sum_{r=1}^{p}(\oplus v_r')\right)a_1 u = \theta,$$

$u, v_1, \cdots, v_p, v_1', \cdots, v_p'$ are independent over θ by Lemma 2.10. Thus $v_1', \cdots, v_p'$ satisfy all the requirements of the lemma.

LEMMA 6.10. *Let $a_1 \geqq a_2 \geqq \cdots$ be an infinite set of elements, and let c satisfy $\theta \leqq c \leqq a_1$, where $\theta = \prod_n a_n$. Suppose further that $c \propto a_n$ for $n = 1, 2, \cdots$. Then there exists an element $c' \leqq a_1$ such that $c \sim c'$, $cc' = \theta$, and $(c \oplus c') \propto a_n$ for $n = 1, 2, \cdots$.*

Proof. Define $c_t = [ca_t - ca_{t+1}]$ for $t = 1, 2, \cdots$; then $ca_t = ca_{t+1} \oplus c_t$, and

$$c = ca_1 = \sum_{t=1}^{\infty} (\oplus\ c_t) \oplus \prod_{t=1}^{\infty} (ca_t)$$

$$= \sum_{t=1}^{\infty} (\oplus\ c_t) + \theta = \sum_{t=1}^{\infty} (\oplus\ c_t).$$

Suppose that $c_{r_1 r_2 \cdots r_n t}$ have been defined, for all $1 \leqq r_1 < r_2 < \cdots < r_n < t < \infty$ with $1 \leqq r_n < p$ for some fixed $p = 1, 2, \cdots$ (n taking all values possible), in such a way that the following conditions are satisfied:

$(\alpha)_p$ c_t, $t = 1, 2, \cdots$; $c_{r_1 r_2 \cdots r_n t}$, $1 \leqq r_1 < r_2 < \cdots < r_n < t$, for $r_n < p$, are independent over θ,

$(\beta)_p$ If we set

$$c'_{r_1 r_2 \cdots r_n} = \sum_{t=r_n+1}^{\infty} (\oplus\ c_{r_1 r_2 \cdots r_n t}),$$

then

$$c_{r_1 r_2 \cdots r_n t} = [c'_{r_1 r_2 \cdots r_n} a_t - c'_{r_1 r_2 \cdots r_n} a_{t+1}],$$

and

$$c_{r_1 r_2 \cdots r_n} \sim c'_{r_1 r_2 \cdots r_n}, \quad 1 \leqq r_1 < r_2 < \cdots < r_n < t < \infty, \ r_n < p.$$

$$(\gamma)_p \qquad c \sim \sum_{t=p}^{\infty} (\oplus\ c_t) \oplus \sum_{t=p}^{\infty} (\oplus\ c_{r_1 r_2 \cdots r_n t}),$$

where in the last summation $r_1, r_2, \cdots, r_n$ take on all possible values with $r_n < p$.

Then $c \propto a_{p+1}$, and

$$c \sim \left(\sum_{t=p+1}^{\infty} (\oplus\ c_t) \oplus \sum_{t=p+1, r_n < p}^{\infty} (\oplus\ c_{r_1 r_2 \cdots r_n t}) \right)$$

$$\oplus \left(c_p \oplus \sum_{r_n < p} (\oplus\ c_{r_1 r_2 \cdots r_n p}) \right);$$

and Lemma 6.9 secures the existence of elements c_p', $c'_{r_1r_2\cdots r_np}$, $(1\leqq r_1<r_2<\cdots<r_n<p)$, such that $\theta\leqq c_p'\leqq a_{p+1}$, $\theta\leqq c_{r_1r_2\cdots r_np}\leqq a_{p+1}$, $c_p\sim c_p'$, $c_{r_1r_2\cdots r_np}\sim c'_{r_1r_2\cdots r_np}$, and c_t, $(t=1, 2, \cdots)$, $c_{r_1r_2\cdots r_nt}$, $(1\leqq r_1<r_2<\cdots<r_n<t<\infty$, $r_n<p)$, c_p', $c'_{r_1r_2\cdots r_np}$, $(1\leqq r_1<r_2<\cdots<r_n<p)$, are independent over θ. Hence if we define $c_{pt}=[c_p'a_t-c_p'a_{t+1}]$ and $c_{r_1r_2\cdots r_npt}=[c'_{r_1r_2\cdots r_np}a_t-c'_{r_1r_2\cdots r_np}a_{t+1}]$, then $(\alpha)_{p+1}$, $(\beta)_{p+1}$, and $(\gamma)_{p+1}$ will be satisfied. Since $(\alpha)_1$, $(\beta)_1$, and $(\gamma)_1$ are trivially satisfied, it follows that we can define the $c_{r_1r_2\cdots r_nt}$, (for all $1\leqq r_1<r_2<\cdots<r_n<t<\infty$), so that $(\alpha)_p$, $(\beta)_p$, and $(\gamma)_p$ are satisfied for all $p=1, 2, \cdots$. Then $c_1, c_2, \cdots, c_1', c_2', \cdots$ are independent over θ, and $c_n\sim c_n'$ for $n=1, 2, \cdots$. Lemmas 2.3 and 4.2 now imply that $c=\sum_{n=1}^{\infty}(\oplus c_n)\sim\sum_{n=1}^{\infty}(\oplus c_n')$. If we set $c'=\sum_{n=1}^{\infty}(\oplus c_n')$, we will have $c\sim c'$, $cc'=\theta$, and $(c\oplus c')\leqq a_1$.

Finally, since $c\propto a_p$, we have $c\sim c^p$ for some c^p with $\theta\leqq c^p\leqq a_p$ by Lemma 6.6. Applying the reasoning of the preceding paragraph to c^p and $a_p\geqq a_{p+1}\geqq\cdots$ we obtain $c^{p\prime}$ such that $c^p\sim c^{p\prime}$ and $(c^p\oplus c^{p\prime})\leqq a_p$. Then $c'\sim c$, $c\sim c^p$, $c^p\sim c^{p\prime}$, imply, by Theorem 5.1, $c'\sim c^{p\prime}$ and hence, by Lemma 6.3, $(c\oplus c')\sim(c^p\oplus c^{p\prime})$. Lemma 6.5 (III) now implies that $(c\oplus c')\propto a_p$ for all $p=1, 2, \cdots$, and c' satisfies all the requirements of the lemma.

Lemma 6.11. *The hypotheses of Lemma* 6.10 *imply* $c=\theta$.

Proof. Suppose that $\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_{2^p}$ have been defined for some fixed $p=0, 1, \cdots$ in such a way that, if we write $c_{(p)}$ for $\bar{c}_1+\cdots+\bar{c}_p$, then

$(\lambda)_p$ $c\sim\bar{c}_r$ for $r=1, \cdots, 2^p$;

$(\mu)_p$ $\bar{c}_r$, $r=1, \cdots, 2^p$, are independent over θ;

$(\nu)_p$ $c_{(p)}\leqq a_1$ and $c_{(p)}\propto a_n$ for $n=1, 2, \cdots$.

Then there exists, by Lemma 6.10, an element $c'\leqq a_1$ such that $c_{(p)}\sim c'$, $c_{(p)}c'=\theta$, and $(c_{(p)}\oplus c')\propto a_n$ for $n=1, 2, \cdots$. Let T be a perspective mapping of $L(\theta, c_{(p)})$ on $L(\theta, c')$, and let $\bar{c}_{2^p+r}=T(\bar{c}_r)$ for $r=1, \cdots, 2^p$. Then $(\lambda)_{p+1}$, $(\mu)_{p+1}$, and $(\nu)_{p+1}$ will be satisfied. Since we can define $\bar{c}_1=c$ to satisfy $(\lambda)_0$, $(\mu)_0$, and $(\nu)_0$, it follows that we can define, by induction on p, an infinite sequence $\bar{c}_n$, $(n=1,2, \cdots)$, satisfying $(\lambda)_p$, $(\mu)_p$, and $(\nu)_p$, for all $p=0, 1, \cdots$. Then we have $\bar{c}_1, \bar{c}_2, \cdots$ independent over θ and $\bar{c}_n\sim\bar{c}_{n+1}$ (by Theorem 5.1 since $\bar{c}_n\sim c$, $c\sim\bar{c}_{n+1}$); hence $\bar{c}_1=\theta$ by Lemma 4.3. Since $c=\bar{c}_1$ we have $c=\theta$. This proves the lemma.

Lemma 6.12. *Without the condition* $c\leqq a_1$, *the remaining hypotheses of Lemma* 6.10 *imply* $c=\theta$.

Proof. $\theta\leqq ca_1$, $c\propto a_1$, imply, by Lemma 6.6, the existence of a c_1 with $\theta\leqq c_1\leqq a_1$, and $c\sim c_1$. By Lemma 6.5 (III), c_1 and $a_1\geqq a_2\geqq\cdots$ satisfy the hypotheses of Lemma 6.10; hence Lemma 6.11 implies $c_1=\theta$. Since $c\sim c_1=\theta$ and $\theta\leqq c$, Lemma 4.4 gives $c=\theta$. This proves the lemma.

LEMMA 6.13. *If θ is defined and $c\propto(a\oplus b)$, there exists a decomposition $c=c_1\oplus c_2$ with $c_1\propto a$, $c_2\propto b$.*

Proof. By Lemma 6.6, $c\sim u$, $\theta\leqq u\leqq(a\oplus b)$. Let $u_1=au$, define $u_1'=[u-u_1]$, and let $u_2=(a+u_1')b$; then $u=u_1\oplus u_1'$, and $u_2\sim u_1'$ with axis a, for

$$u_1'+a=(u_1'+a)(b+a)=(a+u_1')b+a=u_2+a,$$

hence (i) is satisfied; and

$$u_1'a=u_1'ua=u_1'u_1=\theta=u_2ba=u_2a,$$

hence (ii) is satisfied.

Now let $c_1=T^{-1}(u_1)$ and $c_2=T^{-1}(u_1')$; then $c=c_1\oplus c_2$, $c_1\sim u_1\leqq a$, and $c_2\sim u_2\leqq b$ (by Theorem 5.1, since $c_2\sim u_1'$ and $u_1'\sim u_2$). This proves the lemma.

LEMMA 6.14. *If $a_1\geqq a_2\geqq\cdots$ and $c\propto a_n$ for $n=1, 2, \cdots$, then $c\propto\prod_n a_n$.*

Proof. Let $\bar{a}=\prod_n a_n$, $\theta=c\bar{a}$. Apply Lemma 5.1 to $\bar{a}$ and $a_1\geqq a_2\geqq\cdots$ to obtain $a_n=a_n'\oplus\bar{a}$ with $a_1'\geqq a_2'\geqq\cdots$. Then $\prod_n a_n'=(\prod_n a_n')\bar{a}a_1'=\theta$.

Let $c_0=c$ and $\bar{a}_0=\bar{a}$. Suppose c_r, c_r', $\bar{a}_r$, $\bar{a}_r'$, have been defined for $1\leqq r<p$ and for some $p=1, 2, \cdots$ in such a way that the following conditions are satisfied:

$(\alpha)_p$ $c_{r-1}=c_r\oplus c_r'$, $\bar{a}_{r-1}=\bar{a}_r\oplus\bar{a}_r'$, $c_r'\sim\bar{a}_r'$, for $1\leqq r<p$.

$(\beta)_p$ $c_{p-1}\propto a_{p-1}'$, $(p>1)$, and $c_{p-1}\propto(a_n'+\bar{a}_{p-1})$, for $n=1, 2, \cdots$.

Then, since $c_{p-1}\propto(a_p'+\bar{a}_p)$, we can define c_p, c_p', $\bar{a}_p'$, by Lemma 6.13, so that

$$c_{p-1}=c_p\oplus c_p', \qquad c_p\propto a_p', \qquad c_p'\sim\bar{a}_p'\leqq\bar{a}_{p-1}.$$

Now define $\bar{a}_p=[\bar{a}_{p-1}-\bar{a}_p']$. Then, by the use of Lemma 6.8,

$$\bar{a}_{p-1}=\bar{a}_p\oplus\bar{a}_p', \qquad c_p\propto(a_n'\oplus\bar{a}_p), \quad \text{for} \quad n=1, 2, \cdots.$$

Thus $(\alpha)_{p+1}$, $(\beta)_{p+1}$ are satisfied. Since $(\alpha)_1$ and $(\beta)_1$ are satisfied by c_0, $\bar{a}_0$, it follows that we can define by induction c_r, c_r', $\bar{a}_r$, $\bar{a}_r'$, for $r=1, 2, \cdots$, to satisfy $(\alpha)_p$ and $(\beta)_p$ for all $p=1, 2, \cdots$.

By Lemma 2.11

$$c=\sum_{n=1}^{\infty}(\oplus c_n')\oplus\prod_{n=1}^{\infty}c_n, \qquad \bar{a}=\sum_{n=1}^{\infty}(\oplus\bar{a}_n')\oplus\prod_{n=1}^{\infty}\bar{a}_n.$$

Since

$$\prod_{n=1}^{\infty}a_n'=\theta\leqq\prod_{n=1}^{\infty}c_n\leqq c_r\propto a_r' \quad \text{for} \quad r=1, 2, \cdots,$$

Lemma 6.12 implies $\prod_{n=1}^{\infty} c_n = \theta$. Hence $c = \sum_{n=1}^{\infty}(\oplus c_n')$. Since

$$c\left(\sum_{n=1}^{\infty}(\oplus\, \bar{a}_n')\right) = c\bar{a}\left(\sum_{n=1}^{\infty}(\oplus\, \bar{a}_n')\right) = \theta,$$

Lemma 2.10 implies c_n', a_n', $(n=1, 2, \cdots)$, are independent. Lemmas 2.3 and 4.2 now give

$$c = \sum_{n=1}^{\infty}(\oplus\, c_n') \sim \sum_{n=1}^{\infty}(\oplus\, \bar{a}_n) \leqq \bar{a} = \prod_{n=1}^{\infty} a_n.$$

Hence $c \propto \prod_n a_n$, which proves the lemma.

LEMMA 6.15. *If θ is defined, and if*

$$a \oplus a' = b \oplus b',$$

then $b \propto a$ implies $a' \propto b'$.

Proof. Suppose $b \sim a_1 \leqq a$, and define $a_1' = [a - a_1]$. Then $a_1 \oplus (a_1' \oplus a') = b \oplus b'$, and Lemma 6.2 implies $(a_1' \oplus a') \sim b'$; hence by Lemma 6.5 (II) $a' \propto b'$.

LEMMA 6.16. *If $a_1 \leqq a_2 \leqq \cdots$ and if $a_n \propto c$ for $n=1, 2, \cdots$, then $\sum_n a_n \propto c$.*

Proof. Let $a_1 c = \theta$, and define $u_1 = a_1$, $u_n = [a_n - a_{n-1}]$ for $n=2, 3, \cdots$; then $a_n = \sum_{r=1}^{n} u_r$, $(n=1, 2, \cdots)$, and $u_1, u_2, \cdots$ are independent over θ by Lemma 2.6, since $u_{n+1}(u_1 + \cdots + u_n) = [a_{n+1} - a_n]a_n = \theta$ for $n=1, 2, \cdots$. Set $b_n = \sum_{r=n+1}^{\infty}(\oplus u_r)$ for $n=0, 1, \cdots$; then $b_0 \geqq b_1 \geqq \cdots$, and

$$b_0 = \sum_{r=1}^{\infty}(\oplus\, u_r) = a_n \oplus b_n, \qquad n = 1, 2, \cdots.$$

Hence $b_0 \geqq \sum_{n=1}^{\infty} a_n$. Since $u_n \leqq a_n$ we also have $b_0 = \sum_{n=1}^{\infty} u_n \leqq \sum_{n=1}^{\infty} a_n$; thus $b_0 = \sum_{n=1}^{\infty} a_n$. Now define $c' = [(b_0 + c) - c]$ and $b_0' = [(b_0 + c) - b_0]$. Then

$$c \oplus c' = b_0 \oplus b_0' = a_n \oplus (b_n \oplus b_0')$$

for $n=1, 2, \cdots$, and Lemma 6.15 implies $c' \propto (b_n \oplus b_0')$ for $n=1, 2, \cdots$. Applying Lemma 6.14 to c' and $(b_1 + b_0') \geqq (b_2 + b_0') \geqq \cdots$, we obtain

$$c' \propto \prod_n (b_n + b_0') = \left(\prod_n b_n\right) + b_0' = \theta + b_0' = b_0'.$$

Lemma 6.15 now implies $\sum_n a_n = b_0 \propto c$. This proves the lemma.

DEFINITION 6.2. *If $a_1, a_2, \cdots$ is an infinite sequence, we define*

$$\limsup a_n = \prod_p \left(\sum_{n=p}^{\infty} a_n\right), \qquad \liminf a_n = \sum_p \left(\prod_{n=p}^{\infty} a_n\right).$$

The sequence is called convergent if $\lim\sup a_n = \lim\inf a_n$, *and for a convergent sequence we define* $\lim a_n = \lim\sup a_n = \lim\inf a_n$.

LEMMA 6.17. *If* $a_1 \leqq a_2 \leqq \cdots$ $(a_1 \geqq a_2 \geqq \cdots)$ *then* $\lim a_n$ *is defined and is equal to* $\sum_n a_n$ $(\prod_n a_n)$.

Proof.

$$\lim\sup a_n = \prod_{p=1}^{\infty}\left(\sum_{n=p}^{\infty} a_n\right) = \prod_{p=1}^{\infty}\left(\sum_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} a_n$$

$$\left(= \prod_{p=1}^{\infty}(a_p) = \prod_n a_n\right);$$

$$\lim\inf a_n = \sum_{p=1}^{\infty}\left(\prod_{n=p}^{\infty} a_n\right) = \sum_{p=1}^{\infty}(a_p) = \sum_n a_n$$

$$\left(= \sum_{p=1}^{\infty}\left(\prod_{n=1}^{\infty} a_n\right) = \prod_{n=1}^{\infty} a_n\right).$$

Hence $\lim\sup a_n = \lim\inf a_n = \sum_n a_n$ $(= \prod_n a_n)$ and the lemma follows from Definition 6.2.

THEOREM 6.1. CONTINUITY OF PERSPECTIVITY. *If* $a_1, a_2, \cdots$ *and* $b_1, b_2, \cdots$ *are convergent sequences with* $\lim a_n = \bar{a}$ *and* $\lim b_n = \bar{b}$, *then* $a_n \sim b_n$ *for* $n = 1, 2, \cdots$ *implies* $\bar{a} \sim \bar{b}$.

Proof. For every fixed $p = 1, 2, \cdots$ we have

$$\left(\prod_{n=p}^{\infty} a_n\right) \leqq a_r \sim b_r \leqq \left(\sum_{n=r}^{\infty} b_n\right), \qquad r = p, p+1, \cdots,$$

and Lemma 6.14 implies $(\prod_{n=p}^{\infty} a_n) \propto \prod_{r=p}^{\infty}(\sum_{n=r}^{\infty} b_n) = \bar{b}$. Lemma 6.16 gives $\bar{a} = \sum_{p=1}^{\infty}(\prod_{n=p}^{\infty} a_n) \propto \bar{b}$, that is $\bar{a} \propto \bar{b}$. Similarly $\bar{b} \propto \bar{a}$. Then, by Lemma 6.15(IV), $\bar{a} \sim \bar{b}$, which proves the theorem.

COROLLARY. *If* $a_1 \leqq a_2 \leqq \cdots$ *and* $b_1 \leqq b_2 \leqq \cdots$ $(a_1 \geqq a_2 \geqq \cdots$ *and* $b_1 \geqq b_2 \geqq \cdots)$ *then* $a_n \sim b_n$ *for* $n = 1, 2, \cdots$ *implies* $\sum_n a_n \sim \sum_n b_n$ $(\prod_n a_n \sim \prod_n b_n)$.

Proof. By Lemma 6.17 this is a special case of Theorem 6.1.

THEOREM 6.2. ADDITIVITY OF PERSPECTIVITY. *If* θ *is defined and if* a_n, $1 \leqq n < p$, *and* b_n, $1 \leqq n < p$, *are each independent over* θ, *where* p *is finite or infinite, then* $a_n \sim b_n$ *for* $1 \leqq n < p$ *implies*

$$\sum_{n=1}^{p}(\oplus a_n) \sim \sum_{n=1}^{p}(\oplus b_n).$$

Proof. By Lemma 6.4 $\sum_{n=1}^{r}(\oplus a_n) \sim \sum_{n=1}^{r}(\oplus b_n)$ for all $r < p$. If p is finite this proves the theorem, and if p is infinite we have, using the corollary to Theorem 6.1,

$$\sum_{n=1}^{\infty} (\oplus\, a_n) = \sum_{r} \left(\sum_{n=1}^{r} (\oplus\, a_n) \right) \sim \sum_{r} \left(\sum_{n=1}^{r} (\oplus\, b_n) \right) = \sum_{n=1}^{\infty} (\oplus\, b_n).$$

This proves the theorem.*

* For the special case of an irreducible geometry (finite dimensional or continuous), all the lemmas and theorems of §6 are easy consequences of the existence of a dimension function, and conversely, some of them are useful in establishing the existence of the dimension function (see C.G. part 1, chaps. 6 and 7). The notion of a convergent sequence is given in an equivalent form by von Neumann, Proceedings of the National Academy of Sciences, vol. 22 (1936), p. 107 (see the definition of lim** given there).

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